A New Look at Generalized Orthogonal Matching Pursuit: Stable Signal Recovery under Measurement Noise

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***Abstract*—Generalized orthogonal matching pursuit (gOMP) is an extension of orthogonal matching pursuit (OMP) algorithm designed to improve the recovery performance of sparse signals. In this paper, we provide a new analysis for the gOMP algorithm for both noiseless and noisy scenarios. We show that if the measurement matrix $\mathbf{\Phi} \in \mathcal{R}^{m \times n}$ satisfies the restricted isometry property (RIP) with $\delta_{7K+N-1} \leq 0.0231$, then gOMP can perfectly recover any K-sparse signal $\mathbf{x} \in \mathcal{R}^n$ from the measurements $\mathbf{y} = \mathbf{\Phi x}$ within $\lceil \frac{6K}{N} \rceil$ iterations (N is the number of indices chosen in each iteration). We also show that if $\mathbf{\Phi}$ satisfies the RIP with $\delta_{11K+N-1} < 0.0627$, then gOMP can perform a stable recovery of K-sparse signal $\mathbf{x}$ from the noisy measurements $\mathbf{y} = \mathbf{\Phi x} + \mathbf{v}$ within $\lceil \frac{10K}{N} \rceil$ iterations. For Gaussian random measurements, the results indicate that the required measurement size is $m = O(K \log(\frac{n}{K}))$, which is much smaller than the existing result $m = O(K^2 \log(\frac{n}{K}))$.**

***Index Terms*—Compressive sensing (CS), sparse recovery, stable recovery, generalized orthogonal matching pursuit (gOMP), restricted isometry property (RIP)**

I. Introduction

Orthogonal matching pursuit (OMP) is a greedy algorithm widely used for solving sparse recovery problems [1]–[6]. The goal of OMP is to recover a K-sparse vector $\mathbf{x} \in \mathcal{R}^n$ from the measurements

$$\mathbf{y} = \mathbf{\Phi x} \tag{1}$$

where $\mathbf{\Phi} \in \mathcal{R}^{m \times n}$ is a measurement matrix. In each iteration, OMP estimates the support (positions of nonzero elements) of $\mathbf{x}$ by adding an index of the column of $\mathbf{\Phi}$ which is maximally correlated with the residual. The vestige of columns in the estimated support is eliminated from the measurements $\mathbf{y}$, yielding an updated residual for the next iteration. While the number of iterations of OMP is usually set to the sparsity level of the underlying signal to be recovered, there has been some attempts to relax this constraint to enhance the recovery performance. In one direction, an approach allowing more iterations than the sparsity level has been suggested [7]–[10]. In another direction, an algorithm selecting multiple indices in each iteration, referred to as generalized OMP (gOMP) [11], OMMP [12], [13] or KOMP [14], has been proposed. Since it is possible to choose more than one support index in each iteration, gOMP is in general terminated in less than K iterations. Also, it has been empirically shown that gOMP is better than OMP algorithm in recovery performance and computational cost [11], [14].

This work was supported by the KCC (Korea Communications Commission), Korea, under the R&D program supervised by the KCA (Korea Communication Agency) (KCA-12-911-01-110) and the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2012R1A2A2A01047510).

TABLE I
gOMP Algorithm

Input:	measurement matrix $\mathbf{\Phi} \in \mathbb{R}^{m \times n}$,	
	measurements $\mathbf{y} \in \mathbb{R}^m$,	
	sparsity level K,	
	number of indices for each selection N.	
Initialize:	iteration count $k = 0$,	
	estimated list $T^0 = \emptyset$,	
	residual vector $\mathbf{r}^0 = \mathbf{y}$.	
While	$\lVert \mathbf{r}^k \rVert_2 > \epsilon$ and $k < m/N$ **do**	
	$k = k + 1$.	
	$\Lambda^k = \arg\max_{\Lambda:\lvert\Lambda\rvert=N} \lVert (\mathbf{\Phi}' \mathbf{r}^{k-1})_\Lambda \rVert_1$.	**(Identification)**
	$T^k = T^{k-1} \cup \Lambda^k$.	**(Augmentation)**
	$\mathbf{x}^k = \arg\min_{supp(\mathbf{u})=T^k} \lVert \mathbf{y} - \mathbf{\Phi u} \rVert_2$.	**(Estimation)**
	$\mathbf{r}^k = \mathbf{y} - \mathbf{\Phi x}^k$.	**(Residual Update)**
End		
Output:	the estimated signal $\mathbf{x}^k$.	

In analyzing the theoretical performance of gOMP, a property so called restricted isometry property (RIP) has been popularly used [11], [12], [14]. A measurement matrix $\mathbf{\Phi}$ is said to satisfy the RIP of order K if there exists a constant δ such that [15]

$$(1-\delta) \lVert \mathbf{x} \rVert_2^2 \leq \lVert \mathbf{\Phi x} \rVert_2^2 \leq (1+\delta) \lVert \mathbf{x} \rVert_2^2 \tag{2}$$

for any K-sparse vector $\mathbf{x}$. In particular, the minimum of all constants δ satisfying (2) is called the isometry constant δ_K. It has been shown that the perfect recovery of any K-sparse signal via gOMP algorithm is guaranteed under [11]

$$\delta_{NK} < \frac{\sqrt{N}}{\sqrt{K} + 3\sqrt{N}}. \tag{3}$$

For Gaussian random measurements, this result implies that the required size of measurements is [16]

$$m = O\left(K^2 \log\left(\frac{n}{K}\right)\right). \tag{4}$$

While the perfect recovery analysis is possible for noiseless scenario, such is not possible for noisy scenario where the measurements are contaminated by a noise vector $\mathbf{v}$ as

$$\mathbf{y} = \mathbf{\Phi x} + \mathbf{v}. \tag{5}$$

To analyze the recovery performance in this scenario, the ℓ_2-norm of recovery distortion has been commonly employed. In [11], it has been shown that under

$$\delta_{NK} < \frac{\sqrt{N}}{\sqrt{K} + 3\sqrt{N}} \quad \text{and} \quad \delta_{N(K+1)} < 1, \tag{6}$$

gOMP generates an estimate $\mathbf{x}^K$ such that

$$\|\mathbf{x}^K - \mathbf{x}\|_2 \le C_K \|\mathbf{v}\|_2 \tag{7}$$

where $C_K = O(\sqrt{K})$.

The main purpose of this paper is to provide improved recovery bounds of gOMP algorithm in both noiseless and noisy scenarios. The primary contributions of this paper are twofold:

1) In noiseless scenario, we show that if the measurement matrix $\mathbf{\Phi}$ satisfies the RIP with

$$\delta_{7K+N-1} \le 0.0231, \tag{8}$$

then gOMP perfectly recovers any K-sparse signal within $\lceil \frac{6K}{N} \rceil$ iterations.

2) In noisy scenario, we show that under

$$\delta_{11K+N-1} \le 0.0627, \tag{9}$$

gOMP can perform a stable recovery of K-sparse signal $\mathbf{x}$ from the noisy measurements $\mathbf{y} = \mathbf{\Phi x} + \mathbf{v}$ within $\lceil \frac{10K}{N} \rceil$ iterations with the recovery distortion satisfying

$$\|\mathbf{x}^{\lceil \frac{10K}{N} \rceil} - \mathbf{x}\|_2 \le C\|\mathbf{v}\|_2 \tag{10}$$

where C is a constant.

In comparison to our previous work in [11], our new results are distinct and important in two aspects. Firstly, in contrast to the previous recovery bounds in (3) and (6), which are expressed as monotonically decreasing functions of K and hence obviously vanish at large K, the proposed bounds in (8) and (9) are constants. When Gaussian random measurements are employed, the proposed bounds imply that the required measurement size is [16]

$$m = O\left(K \log\left(\frac{n}{K}\right)\right), \tag{11}$$

which is clearly better than the previous result $m = O\left(K^2 \log\left(\frac{n}{K}\right)\right)$ [11]. Secondly and perhaps more importantly, while our previous work demonstrates that the reconstruction error (i.e., the ℓ_2-norm of recovery distortion) in noisy scenario depends linearly on $\sqrt{K}\|\mathbf{v}\|_2$ [11], our new result suggests that the reconstruction error is upper bounded by a constant multiple of $\|\mathbf{v}\|_2$, which ensures the stability of gOMP algorithm under noisy scenario.

The remainder of the paper is organized as follows. In Section II, we introduce lemmas and propositions that are useful in our analysis. In Section III, we provide the recovery bound analysis of gOMP algorithm in both the noiseless and noisy scenarios. In Section IV and V, we provide the recovery condition analysis of the gOMP in the noiseless and noisy scenario, respectively, and conclude the paper in Section VI.

We briefly summarize notations used in this paper. For a vector $\mathbf{x} \in \mathcal{R}^n$, $T = supp(\mathbf{x}) = \{i | x_i \neq 0\}$ represents the set of its non-zero positions. $\Omega = \{1, \cdots, n\}$. For a set $S \subseteq \Omega$, $|S|$ denotes the cardinality of S. $T \backslash S$ is the set of all elements contained in T but not in S. $\mathbf{\Phi}_S \in \mathcal{R}^{m \times |S|}$ is a submatrix of $\mathbf{\Phi}$ that only contains columns indexed by S. $\mathbf{\Phi}'_S$ means the transpose of the matrix $\mathbf{\Phi}_S$. $\mathbf{x}_S \in \mathcal{R}^{|S|}$ is an vector which equals $\mathbf{x}$ for elements indexed by S. If $\mathbf{\Phi}_S$ is full column rank, then $\mathbf{\Phi}^\dagger_S = (\mathbf{\Phi}'_S \mathbf{\Phi}_S)^{-1} \mathbf{\Phi}'_S$ is the pseudo-inverse of $\mathbf{\Phi}_S$. $span(\mathbf{\Phi}_S)$ stands for the span of columns in $\mathbf{\Phi}_S$. $\mathbf{P}_S = \mathbf{\Phi}_S \mathbf{\Phi}^\dagger_S$ is the projection onto $span(\mathbf{\Phi}_S)$. $\mathbf{P}^\perp_S = \mathbf{I} - \mathbf{P}_S$ is the projection onto the orthogonal complement of $span(\mathbf{\Phi}_S)$.

II. Preliminaries

In this section, we provide lemmas and propositions that will be used throughout the paper. Let $\Gamma^k = T \backslash T^k$, then Γ^k is the set of remaining (unselected) support indices after k iterations of gOMP (see Fig. 1(a)). Here and in the rest of the the paper, we assume without loss of generality that $\Gamma^k = \{1, \cdots, |\Gamma^k|\}$. Then it is clear that $0 \le |\Gamma^k| \le K$. For example, if $k = 0$, then $T^k = \emptyset$ and $|\Gamma^k| = |T| = K$. Whereas if $T^k \supseteq T$, then $\Gamma^k = \emptyset$ and $|\Gamma^k| = 0$. Also, for notational convenience we assume that $\{x_i\}_{i=1,2,\cdots,|\Gamma^k|}$ are arranged in descending order of their magnitudes, (i.e., $|x_1| \ge |x_2| \ge \cdots \ge |x_{|\Gamma^k|}|$). Now, we define the subset Γ^k_τ of Γ^k as

$$\Gamma^k_\tau = \begin{cases} \emptyset & \tau = 0, \\ \{1, 2, \cdots, 2^{\tau-1}N\} & \tau = 1, 2, \cdots, \lceil \log_2 \frac{|\Gamma^k|}{N} \rceil, \\ \Gamma^k & \tau = \lceil \log_2 \frac{|\Gamma^k|}{N} \rceil + 1. \end{cases} \tag{12}$$

See Fig. 1(b) for the illustration of Γ^k_τ. Note that the last set $\Gamma^k_{\lceil \log_2 \frac{|\Gamma^k|}{N} \rceil + 1}$ does not necessarily have $2^{\lceil \log_2 \frac{|\Gamma^k|}{N} \rceil} N$ elements.[1]

For a given set Γ^k and a constant $\mu \ge 2$, let $L \in \{1, 2, \cdots, \lceil \log_2 \frac{|\Gamma^k|}{N} \rceil + 1\}$ be the minimum positive integer satisfying[2]

$$\|\mathbf{x}_{\Gamma^k \backslash \Gamma^k_0}\|_2^2 < \mu \|\mathbf{x}_{\Gamma^k \backslash \Gamma^k_1}\|_2^2, \tag{13}$$

$$\|\mathbf{x}_{\Gamma^k \backslash \Gamma^k_1}\|_2^2 < \mu \|\mathbf{x}_{\Gamma^k \backslash \Gamma^k_2}\|_2^2, \tag{14}$$

$$\vdots$$

$$\|\mathbf{x}_{\Gamma^k \backslash \Gamma^k_{L-2}}\|_2^2 < \mu \|\mathbf{x}_{\Gamma^k \backslash \Gamma^k_{L-1}}\|_2^2, \tag{15}$$

$$\|\mathbf{x}_{\Gamma^k \backslash \Gamma^k_{L-1}}\|_2^2 \ge \mu \|\mathbf{x}_{\Gamma^k \backslash \Gamma^k_L}\|_2^2. \tag{16}$$

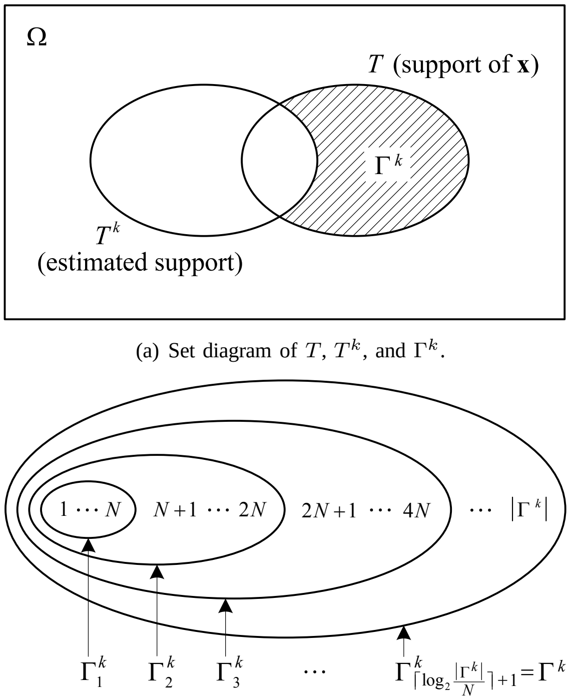
Then, we can easily check the following inequalities.

Lemma 2.1: Γ^k and Γ^k_L satisfy

1) $|\Gamma^k_L| \le |\Gamma^k|$,

[1] For example, if $\Gamma^k = \{1, \cdots, 6N+1\}$, then $\Gamma^k_0 = \emptyset$, $\Gamma^k_1 = \{1, \cdots, N\}$, $\Gamma^k_2 = \{1, \cdots, 2N\}$, $\Gamma^k_3 = \{1, \cdots, 4N\}$, and $\Gamma^k_4 = \{1, \cdots, 6N+1\} = \Gamma^k$ ($|\Gamma^k_4| = 6N + 1 < 8N$).

[2] We note that L is a function of μ and k. Here we use L instead of $L_{\mu,k}$ for notational convenience.



(a) Set diagram of T, T^k, and Γ^k.

(b) Illustration of indices in Γ_τ^k.

Fig. 1. Illustration of sets T, T^k, and Γ^k.

2) $|\Gamma_L^k| \leq 2^{L-1}N$,
3) $|\Gamma^k| \geq 2^{L-2}N$.

The following lemma provides an upper bound of $\|\mathbf{r}^k\|_2^2$.

Lemma 2.2: The residual $\mathbf{r}^k$ of gOMP satisfies

$$\|\mathbf{r}^k\|_2^2 \leq \|\mathbf{\Phi}_{\Gamma^k}\mathbf{x}_{\Gamma^k} + \mathbf{v}\|_2^2.$$

Proof: See Appendix A. ∎

The following proposition provides a lower bound of $\|\mathbf{r}^l\|_2^2 - \|\mathbf{r}^{l+1}\|_2^2$ ($l \geq k$) for gOMP algorithm.

Proposition 2.3: Let $\mathbf{x} \in \mathcal{R}^n$ be any K-sparse vector supported on T, $\mathbf{\Phi} \in \mathcal{R}^{m \times n}$ be a measurement matrix, and $\mathbf{y} = \mathbf{\Phi}\mathbf{x} + \mathbf{v}$ be the noisy measurements. Then for a given Γ^k and any integer $l \geq k$, the residual of gOMP satisfies

$$\begin{aligned}
&\|\mathbf{r}^l\|_2^2 - \|\mathbf{r}^{l+1}\|_2^2 \\
&\geq \frac{1 - \delta_{|\Gamma_\tau^k \cup T^l|}}{\lceil \frac{|\Gamma_\tau^k|}{N} \rceil (1 + \delta_N)} \left(\|\mathbf{r}^l\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k \setminus \Gamma_\tau^k}\mathbf{x}_{\Gamma^k \setminus \Gamma_\tau^k} + \mathbf{v}\|_2^2 \right),
\end{aligned}$$

where $\tau = 0, 1, \cdots, \lceil \log_2 \frac{|\Gamma^k|}{N} \rceil + 1$.

Proof: See Appendix B. ∎

Proposition 2.4 can be viewed as a generalized version of [10, Proposition 3.4]. From this proposition, one can further obtain the following proposition, which will play a key role in our analysis.

Proposition 2.4: For any integer $l' > l$ ($\geq k$) and $\tau \in \{0, 1, \cdots, \lceil \log_2 \frac{|\Gamma^k|}{N} \rceil + 1\}$, the residual $\mathbf{r}^{l'}$ of gOMP satisfies

$$\|\mathbf{r}^{l'}\|_2^2 \leq C_{\tau,l,l'}\|\mathbf{r}^l\|_2^2 + \|\mathbf{\Phi}_{\Gamma^k \setminus \Gamma_\tau^k}\mathbf{x}_{\Gamma^k \setminus \Gamma_\tau^k} + \mathbf{v}\|_2^2,$$

where

$$C_{\tau,l,l'} = \exp\left(-\frac{(l'-l)(1 - \delta_{|\Gamma_\tau^k \cup T^{l'-1}|})}{\lceil \frac{|\Gamma_\tau^k|}{N} \rceil (1 + \delta_N)} \right). \tag{17}$$

Proof: See Appendix C. ∎

III. Recovery of Sparse Signals using gOMP

In this section, we study the performance of gOMP algorithm in recovering sparse signals in both the noiseless and noisy scenarios. We first provide a condition under which gOMP perfectly recovers sparse signals in the noiseless scenario. In this scenario, gOMP can recover any K-sparse signal $\mathbf{x}$ accurately as long as all support indices of the signal $\mathbf{x}$ are chosen. When the iteration loop of gOMP is finished, of course, it is possible that the estimated support T^f contains indices not in T. Even in this case, the final result is unaffected and the original signal $\mathbf{x}$ can be perfectly recovered as long as $T \subseteq T^f$ because

$$\mathbf{x}^f = \arg \min_{supp(\mathbf{x}) = T^f} \|\mathbf{y} - \mathbf{\Phi}\mathbf{x}\|_2$$

and

$$(\mathbf{x}^f)_{T^f} = \mathbf{\Phi}_{T^f}^\dagger \mathbf{y} = \mathbf{\Phi}_{T^f}^\dagger \mathbf{\Phi}_T \mathbf{x}_T = \mathbf{\Phi}_{T^f}^\dagger \mathbf{\Phi}_{T^f} \mathbf{x}_{T^f} = \mathbf{x}_{T^f}.$$

Clearly, it is of importance to find out the iteration number f satisfying $T \subseteq T^f$.

Theorem 3.1 (Exact recovery condition for gOMP): Let $\mathbf{x} \in \mathcal{R}^n$ be any K-sparse vector supported on T and $\mathbf{\Phi} \in \mathcal{R}^{m \times n}$ be the measurement matrix. Then for any positive κ, which is a multiple of $\frac{1}{2}$, and integer $s = (4\kappa + 1)K + N - 1$, there exists $f \leq k + \lceil \frac{4\kappa|\Gamma^k|}{N} \rceil$ such that $T \subseteq T^f$, provided that

$$\frac{1 + \delta_N}{1 - \delta_s} \ln \frac{4(1 + \delta_K)}{1 - \delta_s} \leq \kappa. \tag{18}$$

One can interpret from Theorem 3.1 that after performing k iterations of gOMP, the remaining support indices in Γ^k will be selected within $\lceil \frac{4\kappa|\Gamma^k|}{N} \rceil$ additional iterations. In particular, by applying $k = 0$, one can easily observe that $\Gamma^k = T \setminus T^0 = T$ and $f \leq \lceil \frac{4\kappa K}{N} \rceil$, which implies that gOMP selects all support indices of $\mathbf{x}$ within $\lceil \frac{4\kappa K}{N} \rceil$ iterations.

Corollary 3.2: Let $\mathbf{x} \in \mathcal{R}^n$ be any K-sparse vector and let $\mathbf{\Phi} \in \mathcal{R}^{m \times n}$ be a measurement matrix. Then under $\delta_{7K+N-1} \leq 0.0231$, gOMP perfectly recovers $\mathbf{x}$ from the measurements $\mathbf{y} = \mathbf{\Phi}\mathbf{x}$ within $\lceil \frac{6K}{N} \rceil$ iterations.

Proof: Applying $k = 0$ and $\kappa = 1.5$ in Theorem 3.1, $\Gamma^k = T$, $f \leq \lceil \frac{6K}{N} \rceil$, and $s = 7K + N - 1$. Since $\delta_N \leq \delta_s$ and $\delta_K \leq \delta_s$ in this setting, one can verify that (18) holds under $\delta_{7K+N-1} \leq 0.0231$. ∎

Next, we consider the recovery performance of gOMP in the noisy scenario where the measurements are expressed as $\mathbf{y} = \mathbf{\Phi}\mathbf{x} + \mathbf{v}$. Due to the addition of noise $\mathbf{v}$, the perfect reconstruction of $\mathbf{x}$ cannot be guaranteed, and hence, we use the ℓ_2-norm of recovery distortion as the performance measure.

Theorem 3.3 (Stable recovery under measurement noise): Let $\mathbf{x} \in \mathcal{R}^n$ be any K-sparse vector supported on T,

$\mathbf{\Phi} \in \mathcal{R}^{m\times n}$ be a measurement matrix, and $\mathbf{y} = \mathbf{\Phi}\mathbf{x} + \mathbf{v}$ be the noisy measurements. Then for any positive κ, which is a multiple of $\frac{1}{2}$, and integer $s = (4\kappa+1)K + N - 1$, there exists $f \leq k + \lceil \frac{4\kappa|\Gamma^k|}{N} \rceil$ and constant C and C' such that

$$\|\mathbf{r}^f\|_2 \leq C'\|\mathbf{v}\|_2 \tag{19}$$

and

$$\|\mathbf{x}^f - \mathbf{x}\|_2 \leq C\|\mathbf{v}\|_2, \tag{20}$$

provided that

$$\frac{1+\delta_N}{1-\delta_s} \ln \frac{8(1+\delta_K)}{1-\delta_s} \leq \kappa. \tag{21}$$

One can interpret from Theorem 3.3 that after performing k iterations, gOMP will produce a stable recovery of $\mathbf{x}$ within $\lceil \frac{4\kappa|\Gamma^k|}{N} \rceil$ additional iterations. In particular, by applying $k = 0$, one can obtain that $\Gamma^k = T \backslash T^0 = T$ and $f \leq \lceil \frac{4\kappa K}{N} \rceil$, which implies that the stable recovery of any K-sparse signal $\mathbf{x}$ is guaranteed within $\lceil \frac{4\kappa K}{N} \rceil$ iterations of gOMP.

Corollary 3.4: Let $\mathbf{x} \in \mathcal{R}^n$ be any K-sparse vector, $\mathbf{\Phi} \in \mathcal{R}^{m\times n}$ be a measurement matrix, and $\mathbf{y} = \mathbf{\Phi}\mathbf{x} + \mathbf{v}$ be the noisy measurements. Then under $\delta_{11K+N-1} < 0.0627$, gOMP satisfies

$$\|\mathbf{x}^{\lceil \frac{10K}{N} \rceil} - \mathbf{x}\|_2 \leq C\|\mathbf{v}\|_2$$

where C is a constant.

Proof: Applying $k = 0$ and $\kappa = 2.5$ in Theorem 3.3, we have $\Gamma^k = T$, $k + \lceil \frac{4\kappa|\Gamma^k|}{N} \rceil = \lceil \frac{10K}{N} \rceil$, and $s = 11K + N - 1$. In this case, $\delta_N \leq \delta_s$ and $\delta_K \leq \delta_s$, and hence one can easily show that (21) holds under $\delta_{11K+N-1} \leq 0.0627$. ■

Remark 1 (Stability): We have shown that under the RIP, gOMP can perform a stable recovery of sparse signals under measurement noise. Although we are primarily interested in recovering sparse signals, the result can readily be extended to more general scenarios where the underlying signal is not exactly K-sparse. In essence, our result closes the gap between the theoretical performance of gOMP and other stable recovery approaches (e.g., the BP [17], [18], ROMP [19], and CoSaMP [20]).

Remark 2 (Measurement Size): It is well known that a random measurement matrix $\mathbf{\Phi} \in \mathcal{R}^{m\times n}$ which has i.i.d. entries with Gaussian distribution $N(0, \frac{1}{m})$ obeys the RIP with $\delta_K < \varepsilon$ with overwhelming probability if [16]

$$m = O\left(\frac{K\log\frac{n}{K}}{\varepsilon^2}\right).$$

When the bound $\delta_{NK} < \frac{\sqrt{N}}{\sqrt{K}+3\sqrt{N}}$ [11] is applied, one can check that the gOMP can recover any K-sparse signal given $m = O\left(K^2\log\left(\frac{n}{K}\right)\right)$ Gaussian random measurements of that signal. Whereas, using the proposed bound $\delta_{7K+N-1} \leq 0.0231$, one can obtain $m = O\left(K\log\left(\frac{n}{K}\right)\right)$, which is better than the previous result, in particular for large K.

Remark 3 (Recovery Bounds): In Corollary 3.2 and 3.4, we have used $\kappa = 1.5$ and $\kappa = 2.5$ to obtain recovery bounds for gOMP in noiseless and noisy scenario, respectively. For sure, when different κ is used, one can obtain different recovery

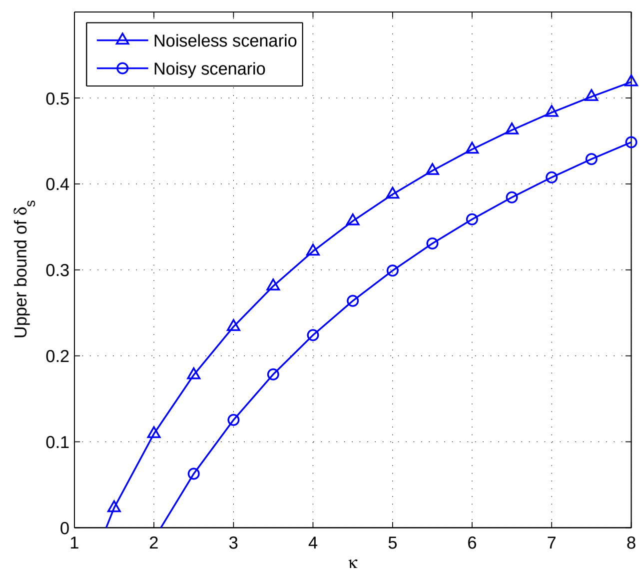


Fig. 2. Recovery bounds of gOMP where $s = (4\kappa+1)K + N - 1$.

bounds. Fig. 2 displays a series of recovery bounds (i.e., the upper bounds of δ_s) as a function of κ for the noiseless and noisy scenarios. We observe that the overall behaviors for two scenarios are analogous. For both cases, the upper bound of δ_s increases monotonically with κ.

Remark 4 (OMP): OMP is a special case of gOMP for $N = 1$. Although the performance of OMP is not better than gOMP, it has been shown that the gap can be reduced when OMP runs more than K iterations. In particular, when the number of indices chosen by OMP is similar to that of gOMP, recovery performances of the two become comparable.[3] In fact, there have been some studies on the recovery performance of OMP when it runs more than K iterations [7]–[10]. In [8], Zhang showed that under $\delta_{31K} < 0.3333$, the stable recovery of K-sparse signals can be achieved in $30K$ iterations. It is interesting to mention that we can also derive recovery bounds for OMP running more than K iterations using Theorem 3.1 and 3.3. Indeed, by applying $N = 1$ in Theorem 3.3, $s = (4\kappa+1)K$ and (21) becomes

$$\frac{1-\delta_1}{1-\delta_s} \ln \frac{8(1+\delta_K)}{1-\delta_s} \leq \kappa. \tag{22}$$

Using relaxations $\delta_1 \leq \delta_s$ and $\delta_K \leq \delta_s$ to (22), one can obtain the recovery bounds of OMP in the noisy scenario. In particular, when $\kappa = 6$ (i.e., $s = 25K$), the maximum number of iterations becomes $24K$, and one can easily check that (22) is guaranteed by $\delta_{25K} \leq 0.3589$, which is better (less restrictive) than $\delta_{31K} < 0.3333$ [8].

[3]In this case, however, the associated computational cost of OMP would be substantial due to the large number of least squares (LS) operations.

IV. Proof of Theorem 3.1

A. Sketch of Proof

Our proof is based on the mathematical induction on $|\Gamma^k|$ (i.e., the number of remaining support indices after k iterations). First, when $|\Gamma^k| = 0$, $T \subseteq T^k$ so that no more iteration is required. Next, we assume that for $|\Gamma^k| = 1, \cdots, \gamma - 1$, if (18) is satisfied then there exists $f \leq k + \lceil \frac{4\kappa|\Gamma^k|}{N} \rceil$ such that $T \subseteq T^f$. Under this assumption, we will show that when $|\Gamma^k| = \gamma$, if (18) is satisfied then there exists

$$f \leq k + \lceil \frac{4\kappa\gamma}{N} \rceil. \tag{23}$$

such that $T \subseteq T^f$.

In the induction step (when $|\Gamma^k| = \gamma$), we first show that a decent amount of support indices in Γ^k can be selected within a specified number of additional iterations. To be specific, let

$$k_i = k + \lceil \kappa (1 + \sum_{\tau=1}^{i} \lceil \frac{|\Gamma^k_\tau|}{N} \rceil) \rceil \quad i = 0, \cdots, L, \tag{24}$$

where κ (> 0) is a multiple of $\frac{1}{2}$, then we show that gOMP algorithm selects more than $2^{L-2}N$ support indices in Γ^k within $k_L - k$ additional iterations (i.e., from the $(k+1)$-th to the k_L-th iteration). In other word, after k_L iterations of gOMP, the number of remaining support indices satisfies

$$|\Gamma^{k_L}| < \gamma - 2^{L-2}N. \tag{25}$$

Thus, by performing a few more iterations (starting from the $(k_L + 1)$-th iteration), all the support indices can be selected. Specifically, since (25) directly implies that $|\Gamma^{k_L}| \leq \gamma - 1$, by the induction hypothesis it requires at most $\lceil \frac{4\kappa|\Gamma^{k_L}|}{N} \rceil$ iterations to select the rest of support indices in Γ^{k_L}.[4] Hence,

$$f \leq k_L + \lceil \frac{4\kappa|\Gamma^{k_L}|}{N} \rceil. \tag{26}$$

The key idea of the induction step is illustrated in Fig. 3.

We next show that the total number of iterations of gOMP is upper bounded by $k + \lceil \frac{4\kappa\gamma}{N} \rceil$. From (26), it is clear that (23) holds true whenever

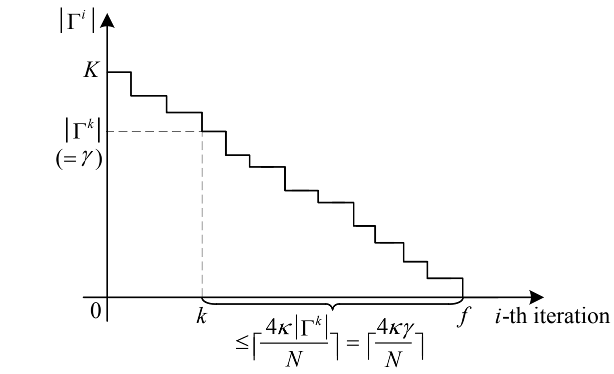
$$k_L + \lceil \frac{4\kappa|\Gamma^{k_L}|}{N} \rceil \leq k + \lceil \frac{4\kappa\gamma}{N} \rceil. \tag{27}$$

Then by using (25), one can easily show that (23) is guaranteed when

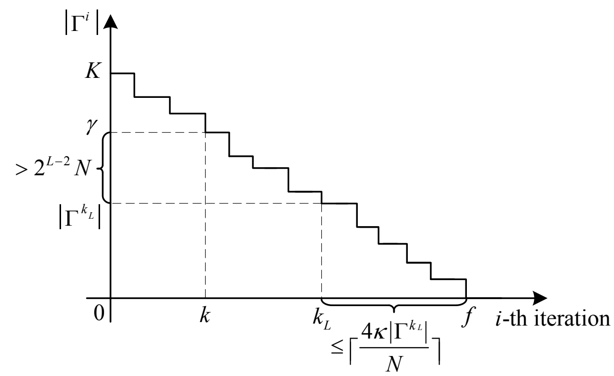
$$k_L + \lceil \frac{4\kappa(\gamma - 2^{L-2}N)}{N} \rceil \leq k + \lceil \frac{4\kappa\gamma}{N} \rceil. \tag{28}$$

Now, in proving $f \leq k + \lceil \frac{4\kappa\gamma}{N} \rceil$ in the induction step, what remains is the verifications of (25) and (28).

[4]In the induction step, we have already assumed that when $|\Gamma^k| \leq \gamma - 1$, it requires $\lceil \frac{4\kappa|\Gamma^k|}{N} \rceil$ maximal iterations to select all indices in Γ^k.



(a) The goal of induction step is to show that $f \leq k + \lceil \frac{4\kappa|\Gamma^k|}{N} \rceil$ holds when $|\Gamma^k| = \gamma$ (i.e., $f \leq k + \lceil \frac{4\kappa\gamma}{N} \rceil$).



(b) The two key ingredients in the proof of $f \leq k + \lceil \frac{4\kappa\gamma}{N} \rceil$ are 1) $|\Gamma^{k_L}| < \gamma - 2^{L-2}N$ and 2) $f \leq k_L + \lceil \frac{4\kappa|\Gamma^{k_L}|}{N} \rceil$.

Fig. 3. Illustration of the induction step.

B. Proof of (25)

Before we proceed, we briefly explain the key steps for proving (25). Consider $\mathbf{x}_{\Gamma^{k_L}}$ and $\mathbf{x}_{\Gamma^k \setminus \Gamma^k_{L-1}}$, which are two truncated vectors of $\mathbf{x}_{\Gamma^k}$. From the definition of Γ^k_τ in (12), $\Gamma^k_{L-1} = \{1, 2, \cdots, 2^{L-2}N\}$ and $\Gamma^k \setminus \Gamma^k_{L-1} = \{2^{L-2}N + 1, \cdots, \gamma\}$, which implies that $|\Gamma^k \setminus \Gamma^k_{L-1}| = \gamma - 2^{L-2}N$ and hence an alternative form of (25) is

$$|\Gamma^{k_L}| < |\Gamma^k \setminus \Gamma^k_{L-1}|. \tag{29}$$

Further, recalling that $\{x_i\}_{i=1,2,\cdots,\gamma}$ are arranged in descending order of their magnitudes (i.e., $|x_1| \geq |x_2| \geq \cdots \geq |x_\gamma|$), it is clear that $\mathbf{x}_{\Gamma^k \setminus \Gamma^k_{L-1}}$ consists of $\gamma - 2^{L-2}N$ most non-significant elements in $\mathbf{x}_{\Gamma^k}$. Therefore, the sufficient condition of (29) is

$$\|\mathbf{x}_{\Gamma^{k_L}}\|_2^2 < \|\mathbf{x}_{\Gamma^k \setminus \Gamma^k_{L-1}}\|_2^2. \tag{30}$$

Let B_u and B_ℓ denote the upper bound of $\|\mathbf{x}_{\Gamma^{k_L}}\|_2^2$ and the lower bound of $\|\mathbf{x}_{\Gamma^k \setminus \Gamma^k_{L-1}}\|_2^2$, respectively. That is,

$$\begin{aligned} \|\mathbf{x}_{\Gamma^{k_L}}\|_2^2 &\leq B_u, \\ \|\mathbf{x}_{\Gamma^k \setminus \Gamma^k_{L-1}}\|_2^2 &> B_\ell. \end{aligned}$$

Then, one can easily check that (30) holds whenever $B_u \leq B_\ell$. Also, since (30) is a sufficient condition of (29) and (29) is equivalent to (25), one can conclude that (25) can be guaranteed under $B_u \leq B_\ell$.

Let's first find out the upper bound B_u of $\|\mathbf{x}_{\Gamma^{k_L}}\|_2^2$. Observe that

$$\begin{aligned}\|\mathbf{r}^{k_L}\|_2^2 &= \|\mathbf{y}-\mathbf{\Phi}\mathbf{x}^{k_L}\|_2^2 &(31)\\ &= \|\mathbf{\Phi}(\mathbf{x}-\mathbf{x}^{k_L})\|_2^2 &(32)\\ &\geq (1-\delta_{|T\cup T^{k_L}|})\|\mathbf{x}-\mathbf{x}^{k_L}\|_2^2 &(33)\\ &\geq (1-\delta_{|T\cup T^{k_L}|})\|\mathbf{x}_{\Gamma^{k_L}}\|_2^2, &(34)\end{aligned}$$

where (32) is due to $\mathbf{y}=\mathbf{\Phi}\mathbf{x}$ in the noiseless scenario and (33) is from the RIP ($\|\mathbf{x}-\mathbf{x}^{k_L}\|_0 \leq |T\cup T^{k_L}|$). Thus,

$$B_u = \frac{\|\mathbf{r}^{k_L}\|_2^2}{1-\delta_{|T\cup T^{k_L}|}}. \quad (35)$$

We next find out a lower bound B_ℓ of $\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2^2$. By applying Proposition 2.4 to $\mathbf{r}^{k_1},\mathbf{r}^{k_2},\cdots,\mathbf{r}^{k_L}$, we have

$$\begin{aligned}\|\mathbf{r}^{k_1}\|_2^2 &\leq C_{1,k_0,k_1}\|\mathbf{r}^{k}\|_2^2+(1+\delta_\gamma)\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_1}\|_2^2, &(36)\\ \|\mathbf{r}^{k_2}\|_2^2 &\leq C_{2,k_1,k_2}\|\mathbf{r}^{k_1}\|_2^2+(1+\delta_\gamma)\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_2}\|_2^2, &(37)\\ &\vdots\\ \|\mathbf{r}^{k_L}\|_2^2 &\leq C_{L,k_{L-1},k_L}\|\mathbf{r}^{k_{L-1}}\|_2^2+(1+\delta_\gamma)\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_L}\|_2^2, &(38)\end{aligned}$$

where we used the facts that $\mathbf{v}=\mathbf{0}$ in the noiseless scenario and that for $\tau=1,\cdots,L$, $|\Gamma^k\setminus\Gamma^k_\tau|\leq|\Gamma^k|=\gamma$ and hence

$$\|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau}+\mathbf{v}\|_2^2 \leq (1+\delta_\gamma)\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau}\|_2^2. \quad (39)$$

From (17) and (24), we have

$$C_{i,k_{i-1},k_i} = \exp\left(-\frac{(k_i-k_{i-1})(1-\delta_{|\Gamma^k_i\cup T^{k_{i-1}}|})}{\lceil\frac{|\Gamma^k_i|}{N}\rceil(1+\delta_N)}\right), \quad i=1,\cdots,L. \quad (40)$$

Since $k_i-k_{i-1}\geq\kappa\lceil\frac{|\Gamma^k_i|}{N}\rceil$ holds true for $i=1,\cdots,L$,[5] we further have

$$C_{i,k_{i-1},k_i} \leq \exp\left(-\kappa\frac{1-\delta_{|\Gamma^k_i\cup T^{k_{i-1}}|}}{1+\delta_N}\right),\ i=1,\cdots,L. \quad (41)$$

Since $\Gamma^k_i\subseteq\Gamma^k_L$ and $T^{k_{i-1}}\subseteq T^{k_{L-1}}$, $\Gamma^k_i\cup T^{k_{i-1}}\subseteq\Gamma^k_L\cup T^{k_{L-1}}$, $|\Gamma^k_i\cup T^{k_{i-1}}|\leq|\Gamma^k_L\cup T^{k_{L-1}}|$, and $\delta_{|\Gamma^k_i\cup T^{k_{i-1}}|}\leq\delta_{|\Gamma^k_L\cup T^{k_{L-1}}|}$ for $i=1,\cdots,L$. For notational simplicity, we let $\eta=\exp\big(-\kappa\frac{1-\delta_{|\Gamma^k_L\cup T^{k_{L-1}}|}}{1+\delta_N}\big)$, then (36)–(38) can be rewritten as

$$\begin{aligned}\|\mathbf{r}^{k_1}\|_2^2 &\leq \eta\|\mathbf{r}^{k}\|_2^2+(1+\delta_\gamma)\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_1}\|_2^2, &(42)\\ \|\mathbf{r}^{k_2}\|_2^2 &\leq \eta\|\mathbf{r}^{k_1}\|_2^2+(1+\delta_\gamma)\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_2}\|_2^2, &(43)\\ &\vdots\\ \|\mathbf{r}^{k_L}\|_2^2 &\leq \eta\|\mathbf{r}^{k_{L-1}}\|_2^2+(1+\delta_\gamma)\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_L}\|_2^2. &(44)\end{aligned}$$

[5]From (12), we have $\lceil\frac{|\Gamma^k_\tau|}{N}\rceil=2^{\tau-1}$ for $\tau=1,\cdots,L-1$. As a result, $k_1-k_0=\lceil\kappa(1+\lceil\frac{|\Gamma^k_1|}{N}\rceil)\rceil=\lceil 2\kappa\rceil$, $k_i-k_{i-1}=k+\lceil\kappa(1+\sum_{\tau=1}^{i}2^{\tau-1})\rceil-(k+\lceil\kappa(1+\sum_{\tau=1}^{i-1}2^{\tau-1})\rceil)=\lceil\kappa\cdot 2^i\rceil-\lceil\kappa\cdot 2^{i-1}\rceil$ for $i=2,\cdots,L-1$, and $k_L-k_{L-1}=\lceil\kappa(2^{L-1}+\lceil\frac{|\Gamma^k_L|}{N}\rceil)\rceil-\lceil\kappa\cdot 2^{L-1}\rceil$. Since κ is a multiple of $\frac{1}{2}$, we further have $k_1-k_0=2\kappa\geq\kappa=\kappa\lceil\frac{|\Gamma^k_1|}{N}\rceil$, $k_i-k_{i-1}=\kappa\cdot 2^i-\kappa\cdot 2^{i-1}=\kappa\cdot 2^{i-1}=\kappa\lceil\frac{|\Gamma^k_i|}{N}\rceil$ for $i=2,\cdots,L-1$, and $k_L-k_{L-1}=\kappa\cdot 2^{L-1}+\lceil\kappa\lceil\frac{|\Gamma^k_L|}{N}\rceil\rceil-\kappa\cdot 2^{L-1}=\lceil\kappa\lceil\frac{|\Gamma^k_L|}{N}\rceil\rceil\geq\kappa\lceil\frac{|\Gamma^k_L|}{N}\rceil$. In summary, $k_i-k_{i-1}\geq\kappa\lceil\frac{|\Gamma^k_i|}{N}\rceil$ holds true for $i=1,\cdots,L$.

After some manipulations, one can easily check that

$$\|\mathbf{r}^{k_L}\|_2^2 \leq \eta^L\|\mathbf{r}^{k}\|_2^2+(1+\delta_\gamma)\sum_{\tau=1}^{L}\eta^{L-\tau}\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau}\|_2^2. \quad (45)$$

By applying Lemma 2.2 for $\mathbf{v}=\mathbf{0}$, we have

$$\begin{aligned}\|\mathbf{r}^{k}\|_2^2 &\leq \|\mathbf{\Phi}_{\Gamma^k}\mathbf{x}_{\Gamma^k}\|_2^2 &(46)\\ &\leq (1+\delta_\gamma)\|\mathbf{x}_{\Gamma^k}\|_2^2 &(47)\\ &\leq (1+\delta_\gamma)\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_0}\|_2^2 &(48)\end{aligned}$$

where (47) is from the RIP ($|\Gamma^k|\leq\gamma$) and (48) uses $\Gamma^k_0=\emptyset$. Plugging (48) into (45) yields

$$\|\mathbf{r}^{k_L}\|_2^2 \leq (1+\delta_\gamma)\sum_{\tau=0}^{L}\eta^{L-\tau}\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau}\|_2^2. \quad (49)$$

Also, from the definition of L in (13)–(16), we have

$$\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau}\|_2^2 \leq \mu^{L-1-\tau}\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2^2 \quad (50)$$

for $\tau=0,1,\cdots,L$. Substituting (49) into (50), we have

$$\begin{aligned}\|\mathbf{r}^{k_L}\|_2^2 &\leq \frac{(1+\delta_\gamma)\,\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2^2}{\mu}\sum_{\tau=0}^{L}(\mu\eta)^{L-\tau}\\ &= \frac{(1+\delta_\gamma)\,\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2^2}{\mu}\sum_{\tau=0}^{L}(\mu\eta)^{\tau}\\ &< \frac{(1+\delta_\gamma)\,\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2^2}{\mu}\sum_{\tau=0}^{\infty}(\mu\eta)^{\tau}. &(51)\end{aligned}$$

When $0<\mu\eta<1$, we further have

$$\|\mathbf{r}^{k_L}\|_2^2 < \frac{(1+\delta_\gamma)\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2^2}{\mu(1-\mu\eta)}. \quad (52)$$

Thus, the desired lower bound B_ℓ of $\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2^2$ is

$$B_\ell = \frac{\mu(1-\mu\eta)\|\mathbf{r}^{k_L}\|_2^2}{1+\delta_\gamma}. \quad (53)$$

As mentioned, it suffices to show $B_u\leq B_\ell$ to prove (25). Using (35) and (53), we have

$$\frac{B_u}{B_\ell} = \frac{1+\delta_\gamma}{\mu(1-\mu\eta)(1-\delta_{|T\cup T^{k_L}|})}. \quad (54)$$

Since $0<\mu\eta<1$ and $\mu(1-\mu\eta)=-\frac{1}{\eta}\big(\mu\eta-\frac{1}{2}\big)^2+\frac{1}{4\eta}$, by choosing $\mu\eta=\frac{1}{2}$, $\mu(1-\mu\eta)$ takes the maximum value $\frac{1}{4\eta}$ and

$$\begin{aligned}\frac{B_u}{B_\ell} &= \frac{4\eta(1+\delta_\gamma)}{1-\delta_{|T\cup T^{k_L}|}}\\ &= \frac{4\,(1+\delta_\gamma)}{1-\delta_{|T\cup T^{k_L}|}}\exp\left(-\kappa\frac{1-\delta_{|\Gamma^k_L\cup T^{k_{L-1}}|}}{1+\delta_N}\right). &(55)\end{aligned}$$

From the monotonicity of isometry constant, we have

$$\begin{aligned}\gamma=|\Gamma^k|\leq|T|=K &\Rightarrow \delta_\gamma\leq\delta_K, &(56)\\ |\Gamma^k_L\cup T^{k_{L-1}}|\leq|T\cup T^{k_L}| &\Rightarrow \delta_{|\Gamma^k_L\cup T^{k_{L-1}}|}\leq\delta_{|T\cup T^{k_L}|}. &(57)\end{aligned}$$

Using (56) and (57), (55) can be rewritten as

$$\frac{B_u}{B_\ell} \leq \frac{4(1+\delta_K)}{1-\delta_{|T\cup T^{k_L}|}} \exp\left(-\kappa \frac{1-\delta_{|T\cup T^{k_L}|}}{1+\delta_N}\right). \tag{58}$$

From (58), it is clear that $B_u \leq B_\ell$ under

$$\frac{4(1+\delta_K)}{1-\delta_{|T\cup T^{k_L}|}} \exp\left(-\kappa \frac{1-\delta_{|T\cup T^{k_L}|}}{1+\delta_N}\right) \leq 1. \tag{59}$$

Equivalently,

$$\frac{1+\delta_N}{1-\delta_{|T\cup T^{k_L}|}} \ln \frac{4(1+\delta_K)}{1-\delta_{|T\cup T^{k_L}|}} \leq \kappa. \tag{60}$$

We now investigate the order $|T \cup T^{k_L}|$ of the isometry constant $\delta_{|T\cup T^{k_L}|}$ in (60). First, we observe that

$$\begin{aligned} |T \cup T^{k_L}| &= |T \backslash T^{k_L}| + |T^{k_L}| \\ &= |\Gamma^{k_L}| + |T^{k_L}| \\ &< \gamma - 2^{L-2}N + k_L N, \end{aligned} \tag{61}$$

where (61) is from (25). Noting that $|T \cup T^{k_L}|$ and $\gamma - 2^{L-2}N + k_L N$ are integers, we have

$$\begin{aligned} |T \cup T^{k_L}| &\leq \gamma - 2^{L-2}N + k_L N - 1 && (62) \\ &\leq \gamma - \frac{1}{2}|\Gamma_L^k| + k_L N - 1 && (63) \\ &\leq K + fN - 1 && (64) \\ &\leq K + \lceil \frac{4\kappa K}{N} \rceil N - 1 && (65) \\ &< K + \big(\frac{4\kappa K}{N} + 1\big) N - 1 && (66) \\ &= (4\kappa + 1)K + N - 1, && (67) \end{aligned}$$

where (63) is from Lemma 2.1, (64) is due to $\gamma = |\Gamma^k| \leq |T| = K$ and $k_L \leq f$ (see Fig. 3(b)), and (65) follows from the fact that $f \leq \lceil \frac{4\kappa K}{N} \rceil$. Denoting $s = (4\kappa+1)K + N - 1$ and using (60) in (67), one can show that (25) holds under

$$\frac{1+\delta_N}{1-\delta_s} \ln \frac{4(1+\delta_K)}{1-\delta_s} \leq \kappa.$$

C. Proof of (28)

In this subsection, we show that $k_L + \lceil \frac{4\kappa(\gamma - 2^{L-2}N)}{N} \rceil \leq k + \lceil \frac{4\kappa\gamma}{N} \rceil$. First, from the definition of k_L,

$$\begin{aligned} k_L &= k + \lceil \kappa (1 + \sum_{\tau=1}^{L} \lceil \frac{|\Gamma_\tau^k|}{N} \rceil) \rceil && (68) \\ &= k + \lceil \kappa (1 + \sum_{\tau=1}^{L-1} \lceil \frac{|\Gamma_\tau^k|}{N} \rceil) + \kappa \lceil \frac{|\Gamma_L^k|}{N} \rceil \rceil && (69) \\ &= k + \lceil \kappa (1 + \sum_{\tau=1}^{L-1} 2^{\tau-1}) + \kappa \lceil \frac{|\Gamma_L^k|}{N} \rceil \rceil && (70) \\ &= k + \lceil \kappa \cdot 2^{L-1} + \kappa \lceil \frac{|\Gamma_L^k|}{N} \rceil \rceil, && (71) \end{aligned}$$

where (70) is from the fact that $|\Gamma_\tau^k| = 2^{\tau-1}N$ for $\tau = 1, \cdots, L-1$. Noting that $|\Gamma_L^k| \leq 2^{L-1}N$ (Lemma 2.1) and κ is a multiple of $\frac{1}{2}$, we have

$$\begin{aligned} k_L &\leq k + \lceil \kappa \cdot 2^{L-1} + \kappa \cdot 2^{L-1} \rceil \\ &= k + \kappa \cdot 2^L. \end{aligned} \tag{72}$$

Then it follows that

$$\begin{aligned} & k_L + \lceil \frac{4\kappa(\gamma - 2^{L-2}N)}{N} \rceil \\ &\leq k + \kappa \cdot 2^L + \lceil \frac{4\kappa(\gamma - 2^{L-2}N)}{N} \rceil \\ &= k + \kappa \cdot 2^L + \lceil \frac{4\kappa\gamma}{N} \rceil - \kappa \cdot 2^L \\ &= k + \lceil \frac{4\kappa\gamma}{N} \rceil, \end{aligned} \tag{73}$$

which completes the proof of (28).

V. Proof of Theorem 3.3

A. Sketch Proof of (19)

The proof of (19) is similar to that of Theorem 3.1 in the sense that it is based on the mathematical induction of the number of remaining indices $|\Gamma^k|$ after k iterations. We first consider the case when $|\Gamma^k| = 0$. In this case, all support indices are selected and $T \subseteq T^k$. Since $\mathbf{x}^k = \arg\min_{supp(\mathbf{u})=T^k} \|\mathbf{y} - \mathbf{\Phi u}\|_2$, we have

$$\begin{aligned} \|\mathbf{r}^k\|_2 &= \|\mathbf{y} - \mathbf{\Phi x}^k\|_2 \\ &= \min_{supp(\mathbf{u})=T^k} \|\mathbf{y} - \mathbf{\Phi u}\|_2 \\ &\leq \|\mathbf{y} - \mathbf{\Phi x}\|_2 \\ &= \|\mathbf{v}\|_2. \end{aligned} \tag{74}$$

Next, we assume that for $|\Gamma^k| = 1, 2, \cdots, \gamma - 1$, if (21) is satisfied then there exists $f \leq k + \lceil \frac{4\kappa|\Gamma^k|}{N} \rceil$ such that $\|\mathbf{r}^f\|_2 \leq C'\|\mathbf{v}\|_2$. Under this assumption, we show that when $|\Gamma^k| = \gamma$, if (21) is satisfied then there exists $f \leq k + \lceil \frac{4\kappa\gamma}{N} \rceil$ such that $\|\mathbf{r}^f\|_2 \leq C'\|\mathbf{v}\|_2$.

When $|\Gamma^k| = \gamma$, we show that a decent amount of indices in Γ^k is selected within a specified number of additional iterations. Specifically, we show that gOMP algorithm selects more than $2^{L-2}N$ support indices in Γ^k within $k_L - k$ additional iterations (k_L is defined in (24)). In other words, the number of remaining support indices $|\Gamma^{k_L}|$ after k_L iterations of gOMP satisfies

$$|\Gamma^{k_L}| < \gamma - 2^{L-2}N. \tag{75}$$

Since $|\Gamma^{k_L}| \leq \gamma - 1$, it is clear from the induction hypothesis that there exists $f \leq k_L + \lceil \frac{4\kappa|\Gamma^{k_L}|}{N} \rceil$ such that $\|\mathbf{r}^f\|_2 \leq C'\|\mathbf{v}\|_2$. In other words, within $k_L + \lceil \frac{4\kappa|\Gamma^{k_L}|}{N} \rceil$ iterations, the ℓ_2-norm of gOMP residual fails below a constant multiple of the noise energy. Since the ℓ_2-norm of the residual is a deceasing function of the iteration number ($\|\mathbf{r}^i\|_2 \leq \|\mathbf{r}^j\|_2$ for $i \geq j$), if we show that

$$k_L + \lceil \frac{4\kappa|\Gamma^{k_L}|}{N} \rceil \leq k + \lceil \frac{4\kappa\gamma}{N} \rceil, \tag{76}$$

then we must have $f \leq k + \lceil \frac{4\kappa\gamma}{N} \rceil$ and the induction step is completed. In fact, using (75) and (76), one can easily show that (76) holds whenever

$$k_L + \lceil \frac{4\kappa(\gamma - 2^{L-2}N)}{N} \rceil \leq k + \lceil \frac{4\kappa\gamma}{N} \rceil,$$

Since we already proved this in (73), we can conclude that (76) holds true and hence the induction step is completed. Now,

what remains in the induction step (when $|\Gamma^k| = \gamma$) is the proof of (75), which will be presented in the next subsection.

B. *Proof of* (75)

The proof of (75) is similar to the proof of (25). Instead of directly checking (75), we show that the sufficient condition of (75) is true. To be specific, we show that[6]

$$\|\mathbf{x}_{\Gamma^{k_L}}\|_2^2 < \|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2^2. \tag{77}$$

To do so, we first construct lower and upper bounds of $\|\mathbf{r}^{k_L}\|_2$ and then derive a condition guaranteeing (77).

First, we build a lower bound of $\|\mathbf{r}^{k_L}\|_2^2$. Noting that $\|\mathbf{x} - \mathbf{x}^{k_L}\|_0 \le |T \cup T^{k_L}|$, we have

$$\begin{aligned} \|\mathbf{r}^{k_L}\|_2 &= \|\mathbf{y} - \mathbf{\Phi}\mathbf{x}^{k_L}\|_2 \\ &= \|\mathbf{\Phi}(\mathbf{x} - \mathbf{x}^{k_L}) + \mathbf{v}\|_2 \\ &\ge \|\mathbf{\Phi}(\mathbf{x} - \mathbf{x}^{k_L})\|_2 - \|\mathbf{v}\|_2 \\ &\ge \sqrt{1 - \delta_{|T\cup T^{k_L}|}}\|\mathbf{x} - \mathbf{x}^{k_L}\|_2 - \|\mathbf{v}\|_2 \\ &\ge \sqrt{1 - \delta_{|T\cup T^{k_L}|}}\|\mathbf{x}_{\Gamma^{k_L}}\|_2 - \|\mathbf{v}\|_2. \end{aligned} \tag{78}$$

Next, we find out an upper bound of $\|\mathbf{r}^{k_L}\|_2^2$. By applying Proposition 2.4, we have

$$\begin{aligned} \|\mathbf{r}^{k_1}\|_2^2 &\le C_{1,k_0,k_1}\|\mathbf{r}^k\|_2^2 + \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_1}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_1} + \mathbf{v}\|_2^2, && (79)\\ \|\mathbf{r}^{k_2}\|_2^2 &\le C_{2,k_1,k_2}\|\mathbf{r}^{k_1}\|_2^2 + \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_2}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_2} + \mathbf{v}\|_2^2, && (80)\\ &\vdots \\ \|\mathbf{r}^{k_L}\|_2^2 &\le C_{L,k_{L-1},k_L}\|\mathbf{r}^{k_{L-1}}\|_2^2 + \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_L}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_L} + \mathbf{v}\|_2^2. && (81) \end{aligned}$$

Using (41), we further have

$$\begin{aligned} \|\mathbf{r}^{k_1}\|_2^2 &\le \eta\|\mathbf{r}^k\|_2^2 + \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_1}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_1} + \mathbf{v}\|_2^2, \\ \|\mathbf{r}^{k_2}\|_2^2 &\le \eta\|\mathbf{r}^{k_1}\|_2^2 + \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_1}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_1} + \mathbf{v}\|_2^2, \\ &\vdots \\ \|\mathbf{r}^{k_L}\|_2^2 &\le \eta\|\mathbf{r}^{k_{L-1}}\|_2^2 + \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_L}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_L} + \mathbf{v}\|_2^2, \end{aligned}$$

where $\eta = \exp\big(-\kappa\frac{1-\delta_{|\Gamma^k_L\cup T^{k_{L-1}}|}}{1+\delta_N}\big)$. After some manipulations, we have

$$\|\mathbf{r}^{k_L}\|_2^2 \le \eta^L\|\mathbf{r}^k\|_2^2 + \sum_{\tau=1}^{L}\eta^{L-\tau}\|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2. \tag{82}$$

By applying Lemma 2.2 (i.e., $\|\mathbf{r}^k\|_2^2 \le \|\mathbf{\Phi}_{\Gamma^k}\mathbf{x}_{\Gamma^k} + \mathbf{v}\|_2^2$) and also noting that $\Gamma^k_0 = \emptyset$, (82) becomes

$$\begin{aligned} \|\mathbf{r}^{k_L}\|_2^2 &\le \sum_{\tau=0}^{L}\eta^{L-\tau}\|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2 && (83)\\ &\le \sum_{\tau=0}^{L}2\eta^{L-\tau}(\|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau}\|_2^2 + \|\mathbf{v}\|_2^2) && (84)\\ &\le 2(1+\delta_\gamma)\sum_{\tau=0}^{L}\eta^{L-\tau}\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau}\|_2^2 \\ &\quad + 2\sum_{\tau=0}^{L}\eta^{L-\tau}\|\mathbf{v}\|_2^2 && (85)\\ &< 2(1+\delta_\gamma)\sum_{\tau=0}^{L}\eta^{L-\tau}\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau}\|_2^2 \\ &\quad + \frac{2\|\mathbf{v}\|_2^2}{1-\eta}, && (86) \end{aligned}$$

where (85) uses the RIP ($|\Gamma^k\setminus\Gamma^k_\tau| \le |\Gamma^k| = \gamma$ for $\tau = 0, 1, \cdots, L$) and (86) is due to $\eta = \exp\big(-\kappa\frac{1-\delta_{|\Gamma^k_L\cup T^{k_{L-1}}|}}{1+\delta_N}\big) < 1$ and hence $\sum_{\tau=0}^{L}\eta^{L-\tau} = \sum_{\tau=0}^{L}\eta^\tau < \sum_{\tau=0}^{\infty}\eta^{L-\tau} = \frac{1}{1-\eta}$.

Also, from the definition of L in (13)–(16), we have

$$\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau}\|_2^2 \le \mu^{L-1-\tau}\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2^2 \tag{87}$$

for $\tau = 0, 1, \cdots, L$. Substituting (87) into (86), we have

$$\|\mathbf{r}^{k_L}\|_2^2 \le \frac{2(1+\delta_\gamma)\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2^2}{\mu}\sum_{\tau=0}^{L}(\mu\eta)^{L-\tau} + \frac{2\|\mathbf{v}\|_2^2}{1-\eta}. \tag{88}$$

We choose $\mu\eta = \frac{1}{2}$ so that $\sum_{\tau=0}^{L}(\mu\eta)^{L-\tau} = \sum_{\tau=0}^{L}\frac{1}{2^\tau} < \sum_{\tau=0}^{\infty}\frac{1}{2^\tau} = 2$ and

$$\|\mathbf{r}^{k_L}\|_2^2 < 8\eta\,(1+\delta_\gamma)\,\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2^2 + \frac{2\|\mathbf{v}\|_2^2}{1-\eta}. \tag{89}$$

Hence, the upper bound of $\|\mathbf{r}^{k_L}\|_2$ is

$$\|\mathbf{r}^{k_L}\|_2 < 2\sqrt{2\eta\,(1+\delta_\gamma)}\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2 + \frac{\sqrt{2}}{\sqrt{1-\eta}}\|\mathbf{v}\|_2. \tag{90}$$

Now, using the lower bound in (78) and the upper bound in (90), we have

$$\|\mathbf{x}_{\Gamma^{k_L}}\|_2 < \alpha\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2 + \beta\|\mathbf{v}\|_2 \tag{91}$$

where

$$\alpha = 2\sqrt{\frac{2\eta(1+\delta_\gamma)}{1-\delta_{|T\cup T^{k_L}|}}} \quad\text{and}\quad \beta = \frac{\sqrt{2}+\sqrt{1-\eta}}{\sqrt{(1-\eta)(1-\delta_{|T\cup T^{k_L}|})}}.$$

Recalling that $\eta = \exp\big(-\kappa\frac{1-\delta_{|\Gamma^k_L\cup T^{k_{L-1}}|}}{1+\delta_N}\big)$, one can show that $\alpha \le 1$ is guaranteed by

$$\frac{8(1+\delta_K)}{1-\delta_{|T\cup T^{k_L}|}}\exp\left(-\kappa\frac{1-\delta_{|\Gamma^k_L\cup T^{k_{L-1}}|}}{1+\delta_N}\right) \le 1. \tag{92}$$

Further, noting that $\delta_{|\Gamma^k_L\cup T^{k_{L-1}}|} \le \delta_{|T\cup T^{k_L}|}$ and $|T\cup T^{k_L}| < (4\kappa+1)K + N - 1$ (see (67)), and denoting $s = (4\kappa+1)K + N - 1$, $\alpha \le 1$ holds true if

$$\frac{8(1+\delta_K)}{1-\delta_s}\exp\left(-\kappa\frac{1-\delta_s}{1+\delta_N}\right) \le 1. \tag{93}$$

[6]Recalling that $\{x_i\}_{i=1,2,\cdots,\gamma}$ are arranged in descending order of their magnitudes ($|x_1| \ge |x_2| \ge \cdots \ge |x_\gamma|$), it is clear that $\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}$ consists of $\gamma - 2^{L-2}N$ most non-significant elements in $\mathbf{x}_{\Gamma^k}$, and therefore (77) is a sufficient condition of (75).

Equivalently,

$$\frac{1+\delta_N}{1-\delta_s}\ln\frac{8(1+\delta_K)}{1-\delta_s}\leq\kappa. \tag{94}$$

We consider the case when $\beta\|\mathbf{v}\|_2\leq(1-\alpha)\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2$. In this case, (91) implies that

$$\|\mathbf{x}_{\Gamma^k L}\|_2^2<\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2^2,$$

and hence (75) holds true. The alternative case (i.e., $\beta\|\mathbf{v}\|_2>(1-\alpha)\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2$) is trivial. Indeed, when $\beta\|\mathbf{v}\|_2>(1-\alpha)\|\mathbf{x}_{\Gamma^k\setminus\Gamma^k_{L-1}}\|_2$, (90) implies that

$$\begin{aligned}\|\mathbf{r}^{k_L}\|_2 &< \big(\frac{\beta}{1-\alpha}\sqrt{1-\delta_{|T\cup T^{k_L}|}}-1\big)\|\mathbf{v}\|_2\\ &< \frac{\alpha+\beta-1}{1-\alpha}\|\mathbf{v}\|_2\\ &\leq C'\|\mathbf{v}\|_2.\end{aligned}$$

C. *Proof of* (20)

So far we have shown that (19) holds true under (21) and we are now ready to prove (20). First, we observe that

$$\begin{aligned}\|\mathbf{r}^f\|_2 &= \|\mathbf{y}-\mathbf{\Phi}\mathbf{x}^f\|_2\\ &= \|\mathbf{\Phi}(\mathbf{x}-\mathbf{x}^f)+\mathbf{v}\|_2\\ &\geq \|\mathbf{\Phi}(\mathbf{x}-\mathbf{x}^f)\|_2-\|\mathbf{v}\|_2\\ &\geq \big(1-\delta_{|T\cup T^f|}\big)\,\|\mathbf{x}-\mathbf{x}^f\|_2-\|\mathbf{v}\|_2 \qquad (95)\end{aligned}$$

where (95) is due to the RIP. We now consider the order $|T\cup T^f|$ of the isometry constant $\delta_{|T\cup T^f|}$. Note that

$$\begin{aligned}|T\cup T^f| &= |T\setminus T^f|+|T^f| &(96)\\ &\leq |T\setminus T^{k_L}|+N\lceil\frac{4\kappa K}{N}\rceil &(97)\\ &< |\Gamma^k|-2^{L-2}N+N\lceil\frac{4\kappa K}{N}\rceil, &(98)\end{aligned}$$

where (97) is because $k_L\leq f$ and $f\leq\lceil\frac{4\kappa K}{N}\rceil$, and (98) is due to (75). Since $|T\cup T^f|$ is an integer, we have

$$\begin{aligned}|T\cup T^f| &\leq |\Gamma^k|-2^{L-2}N+N\lceil\frac{4\kappa K}{N}\rceil-1 &(99)\\ &\leq |\Gamma^k|-\frac{1}{2}|\Gamma^k_L|+N\lceil\frac{4\kappa K}{N}\rceil-1 &(100)\\ &\leq K+N\lceil\frac{4\kappa K}{N}\rceil-1 &(101)\\ &< K+N\big(\frac{4\kappa K}{N}+1\big)-1 &(102)\\ &= \big(4\kappa+1\big)K+N-1, &(103)\end{aligned}$$

where (100) is from Lemma 2.1 and (101) is due to $|\Gamma^k|\leq K$. Let $s=(4\kappa+1)K+N-1$, then using (95), we have

$$\|\mathbf{r}^f\|_2\geq(1-\delta_s)\,\|\mathbf{x}-\mathbf{x}^f\|_2-\|\mathbf{v}\|_2. \tag{104}$$

Equivalently,

$$\|\mathbf{x}-\mathbf{x}^f\|_2\leq\frac{\|\mathbf{r}^f\|_2+\|\mathbf{v}\|_2}{1-\delta_s}. \tag{105}$$

Recalling from (19) that

$$\|\mathbf{r}^f\|_2\leq C'\|\mathbf{v}\|_2,$$

we further have

$$\|\mathbf{x}-\mathbf{x}^f\|_2\leq\frac{(1+C')\|\mathbf{v}\|_2}{1-\delta_s}=C\|\mathbf{v}\|_2, \tag{106}$$

which completes the proof of Theorem 3.3.

VI. Conclusion

With the aim to enhance the recovery performance of OMP, gOMP algorithm has been proposed. Unlike OMP selecting one index in each iteration, gOMP chooses multiple indices, which helps to catch more than one support indices in each iteration. While gOMP has been shown to be competitive empirically, little has been known whether it can perform a uniform recovery of sparse signals under a natural RIP. In this paper, we have shown that gOMP performs an exact recovery of any K-sparse signal in noiseless scenario, provided that $\delta_{7K+N-1}\leq 0.0231$. We have also shown that in the noisy scenario gOMP can provide a stable recovery of any K-sparse signal from its perturbed measurements under $\delta_{11K+N-1}<0.0627$. It is worth mentioning that the stability result can be readily extended to the more general scenario where the underlying signals to be recovered are approximately sparse. Also, the presented proof strategy might be useful to obtain an improved recovery bound of greedy algorithms derived from the OMP.

Appendix A
Proof of Lemma 2.2

Proof: $T^k\cap T\subseteq T^k$ implies that

$$\|\mathbf{r}^k\|_2^2=\|\mathbf{P}^\perp_{T^k}\mathbf{y}\|_2^2\leq\|\mathbf{P}^\perp_{T^k\cap T}\mathbf{y}\|_2^2. \tag{107}$$

Also, noting that $\mathbf{P}^\perp_{T^k\cap T}\mathbf{y}$ is the projection of $\mathbf{y}$ onto the orthogonal complement of $span(\mathbf{\Phi}_{T^k\cap T})$,

$$\|\mathbf{P}^\perp_{T^k\cap T}\mathbf{y}\|_2^2=\min_{supp(\mathbf{z})=T^k\cap T}\|\mathbf{y}-\mathbf{\Phi}\mathbf{z}\|_2^2. \tag{108}$$

From (107) and (108), we have

$$\begin{aligned}\|\mathbf{r}^k\|_2^2 &\leq \big\|\mathbf{y}-\mathbf{\Phi}_{T^k\cap T}\mathbf{x}_{T^k\cap T}\big\|_2^2\\ &= \big\|\mathbf{\Phi}_T\mathbf{x}_T+\mathbf{v}-\mathbf{\Phi}_{T^k\cap T}\mathbf{x}_{T^k\cap T}\big\|_2^2\\ &= \|\mathbf{\Phi}_{\Gamma^k}\mathbf{x}_{\Gamma^k}+\mathbf{v}\|_2^2, \qquad (109)\end{aligned}$$

where (109) is from $T\setminus(T^k\cap T)=T\setminus T^k=\Gamma^k$. ■

Appendix B
Proof of Proposition 2.3

The following lemmas are useful in the proof of Proposition 2.3.

Lemma B.1: Let $\mathbf{u},\mathbf{z}\in\mathcal{R}^n$ be two distinct vectors and let $W=supp(\mathbf{u})\cap supp(\mathbf{z})$. Also, let U be the set of N indices corresponding to N most significant elements in $\mathbf{u}$. Then for any integer $N\geq 1$,

$$\langle\mathbf{u},\mathbf{z}\rangle\leq\sqrt{\lceil\frac{|W|}{N}\rceil}\,\|\mathbf{u}_U\|_2\|\mathbf{z}_W\|_2. \tag{110}$$

Proof: We first consider the case $1\leq N\leq|W|$. Without loss of generality, we assume that $W=\{1,2,\cdots,|W|\}$ and

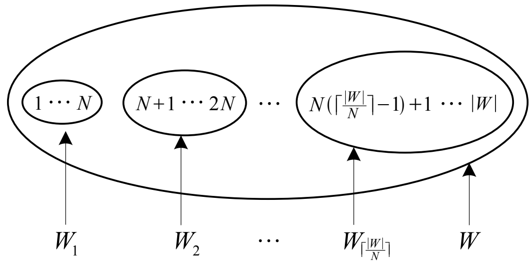


Fig. 4. Illustration of indices in W_i.

that the elements of $\mathbf{u}_W$ are arranged in a descending order of their magnitudes. We define the subset W_i of W as[7]

$$W_i = \begin{cases} \{N(i-1)+1, \cdots, Ni\} & i = 1, \cdots, \lceil \frac{|W|}{N} \rceil - 1, \\ \{N(\lceil \frac{|W|}{N} \rceil - 1)+1, \cdots, |W|\} & i = \lceil \frac{|W|}{N} \rceil. \end{cases} \tag{111}$$

Then it is clear that

$$\begin{aligned} \langle \mathbf{u}, \mathbf{z} \rangle &= \langle \mathbf{u}_W, \mathbf{z}_W \rangle & (112) \\ &\le \sum_i |\langle \mathbf{u}_{W_i}, \mathbf{z}_{W_i} \rangle| & (113) \\ &\le \sum_i \|\mathbf{u}_{W_i}\|_2 \|\mathbf{z}_{W_i}\|_2, & (114) \end{aligned}$$

where (113) is due to the Hölder's inequality. By the definition of U, we have $\|\mathbf{u}_U\|_2 \ge \|\mathbf{u}_{W_1}\|_2 = \max_i \|\mathbf{u}_{W_i}\|_2$ and hence

$$\begin{aligned} \langle \mathbf{u}, \mathbf{z} \rangle &\le \|\mathbf{u}_U\|_2 \sum_i \|\mathbf{z}_{W_i}\|_2 & (115) \\ &\le \|\mathbf{u}_U\|_2 \sqrt{\lceil \frac{|W|}{N} \rceil \sum_i \|\mathbf{z}_{W_i}\|_2^2} & (116) \\ &= \sqrt{\lceil \frac{|W|}{N} \rceil} \|\mathbf{u}_U\|_2 \|\mathbf{z}_W\|_2 & (117) \end{aligned}$$

where (116) follows from the fact that $\sum_{i=1}^d a_i \le \sqrt{d \sum_{i=1}^d a_i^2}$ with $a_i = \|\mathbf{z}_{W_i}\|_2$ and $d = \lceil \frac{|W|}{N} \rceil$.

Next, we consider the alternative case where $N > |W|$. In this case, it is clear that $\sqrt{\lceil \frac{|W|}{N} \rceil} = 1$ and $\|\mathbf{u}_U\|_2 \ge \|\mathbf{u}_W\|_2$, and hence

$$\begin{aligned} \sqrt{\lceil \frac{|W|}{N} \rceil} \|\mathbf{u}_U\|_2 \|\mathbf{z}_W\|_2 &= \|\mathbf{u}_U\|_2 \|\mathbf{z}_W\|_2 \\ &\ge \|\mathbf{u}_W\|_2 \|\mathbf{z}_W\|_2 \\ &\ge \langle \mathbf{u}_W, \mathbf{z}_W \rangle \\ &= \langle \mathbf{u}, \mathbf{z} \rangle, & (118) \end{aligned}$$

which is the desired result. ■

The following lemma characterizes the reduction of residual (in ℓ_2-norm) in the $(l+1)$-th iteration of gOMP.

Lemma B.2: Let $\mathbf{x} \in \mathcal{R}^n$ be any K-sparse vector supported on T and $\mathbf{\Phi} \in \mathcal{R}^{m \times n}$ be the measurement matrix with unit ℓ_2-norm columns. Then, for any integer $l \ge k$, the residual of gOMP satisfies

$$\|\mathbf{r}^l - \mathbf{r}^{l+1}\|_2^2 \ge \frac{1}{1+\delta_N} \|\mathbf{\Phi}'_{\Lambda^{k+1}} \mathbf{r}^l\|_2^2. \tag{119}$$

Proof: Recall that gOMP algorithm orthogonalizes the measurements $\mathbf{y}$ against previously chosen columns of $\mathbf{\Phi}$, yielding the updated residual in each iteration. That is,

$$\mathbf{r}^{l+1} = \mathbf{P}_{T^{l+1}}^{\perp} \mathbf{y}. \tag{120}$$

Since $\mathbf{r}^l = \mathbf{y} - \mathbf{\Phi}\mathbf{x}^l$, we have

$$\mathbf{r}^{l+1} = \mathbf{P}_{T^{k+1}}^{\perp} (\mathbf{r}^l + \mathbf{\Phi}\mathbf{x}^l) = \mathbf{P}_{T^{l+1}}^{\perp} \mathbf{r}^l. \tag{121}$$

where (121) is because $\mathbf{\Phi}\mathbf{x}^l \in span(\mathbf{\Phi}_{T^l})$ and $T^l \subset T^{l+1}$ and hence $\mathbf{P}_{T^{l+1}}^{\perp} \mathbf{\Phi}\mathbf{x}^l = \mathbf{0}$. As a result,

$$\mathbf{r}^l - \mathbf{r}^{l+1} = \mathbf{r}^l - \mathbf{P}_{T^{l+1}}^{\perp} \mathbf{r}^l = \mathbf{P}_{T^{l+1}} \mathbf{r}^l. \tag{122}$$

Noting that $\Lambda^{l+1} \subseteq T^{l+1}$, we have

$$\|\mathbf{r}^l - \mathbf{r}^{l+1}\|_2 = \|\mathbf{P}_{T^{l+1}} \mathbf{r}^l\|_2 \ge \|\mathbf{P}_{\Lambda^{l+1}} \mathbf{r}^l\|_2. \tag{123}$$

Since $\mathbf{P}_{\Lambda^{l+1}} = \mathbf{P}'_{\Lambda^{l+1}} = (\mathbf{\Phi}^{\dagger}_{\Lambda^{l+1}})' \mathbf{\Phi}'_{\Lambda^{l+1}}$, we further have

$$\begin{aligned} \|\mathbf{r}^l - \mathbf{r}^{l+1}\|_2 &\ge \|(\mathbf{\Phi}^{\dagger}_{\Lambda^{l+1}})' \mathbf{\Phi}'_{\Lambda^{l+1}} \mathbf{r}^l\|_2 \\ &\ge \frac{1}{\sqrt{1+\delta_N}} \|\mathbf{\Phi}'_{\Lambda^{l+1}} \mathbf{r}^l\|_2 & (124) \end{aligned}$$

where (124) is due to the fact that the singular values of $\mathbf{\Phi}_{\Lambda^{l+1}}$ lie between $\sqrt{1-\delta_N}$ and $\sqrt{1+\delta_N}$. ■

In the next lemma, we find out a lower bound of $\|\mathbf{\Phi}'_{\Lambda^{l+1}} \mathbf{r}^l\|_2$.

Lemma B.3: Let $\mathbf{x} \in \mathcal{R}^n$ be any K-sparse vector supported on T and $\mathbf{\Phi} \in \mathcal{R}^{m \times n}$ be the measurement matrix with unit ℓ_2-norm columns. Then, for a given set Γ^k and an integer $l \ge k$, the residual of gOMP satisfies

$$\|\mathbf{\Phi}'_{\Lambda^{l+1}} \mathbf{r}^l\|_2^2 \ge \frac{1 - \delta_{|\Gamma_\tau^k \cup T^l|}}{\lceil \frac{|\Gamma_\tau^k|}{N} \rceil} \left(\|\mathbf{r}^l\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k \setminus \Gamma_\tau^k} \mathbf{x}_{\Gamma^k \setminus \Gamma_\tau^k} + \mathbf{v}\|_2^2 \right). \tag{125}$$

Proof: Let $\mathbf{z} \in \mathcal{R}^n$ be the vector satisfying $\mathbf{z}_{T \cap T^k \cup \Gamma_\tau^k} = \mathbf{x}_{T \cap T^k \cup \Gamma_\tau^k}$ and $\mathbf{z}_{\Omega \setminus (T \cap T^k \cup \Gamma_\tau^k)} = \mathbf{0}$. Further, let $\mathbf{u} = \mathbf{\Phi}' \mathbf{r}^l$, $U = \Lambda^{l+1}$, and $W = \Gamma_\tau^k \setminus T^l$.[8] Then using Lemma B.1, we have

$$\begin{aligned} \langle \mathbf{\Phi}' \mathbf{r}^l, \mathbf{z} \rangle &\le \sqrt{\lceil \frac{|\Gamma_\tau^k \setminus T^l|}{N} \rceil} \|\mathbf{\Phi}'_{\Lambda^{l+1}} \mathbf{r}^l\|_2 \|\mathbf{z}_{\Gamma_\tau^k \setminus T^l}\|_2 \\ &\le \sqrt{\lceil \frac{|\Gamma_\tau^k|}{N} \rceil} \|\mathbf{\Phi}'_{\Lambda^{l+1}} \mathbf{r}^l\|_2 \|\mathbf{z}_{\Gamma_\tau^k \setminus T^l}\|_2. & (126) \end{aligned}$$

Since $supp(\mathbf{x}^l) = T^l$ and $supp(\mathbf{\Phi}' \mathbf{r}^l) = \Omega \setminus T^l$, we have $\langle \mathbf{\Phi}' \mathbf{r}^l, \mathbf{x}^l \rangle = \mathbf{0}$ and $\langle \mathbf{\Phi}' \mathbf{r}^l, \mathbf{z} \rangle = \langle \mathbf{\Phi}' \mathbf{r}^l, \mathbf{z} - \mathbf{x}^l \rangle$ and thus

$$\begin{aligned} \|\mathbf{\Phi}'_{\Lambda^{l+1}} \mathbf{r}^l\|_2 &\ge \frac{\langle \mathbf{\Phi}' \mathbf{r}^l, \mathbf{z} \rangle}{\sqrt{\lceil \frac{|\Gamma_\tau^k|}{N} \rceil} \|\mathbf{z}_{\Gamma_\tau^k \setminus T^l}\|_2} \\ &= \frac{\langle \mathbf{\Phi}' \mathbf{r}^l, \mathbf{z} - \mathbf{x}^l \rangle}{\sqrt{\lceil \frac{|\Gamma_\tau^k|}{N} \rceil} \|\mathbf{z}_{\Gamma_\tau^k \setminus T^l}\|_2}. & (127) \end{aligned}$$

[7] We note that when $\lceil \frac{|W|}{N} \rceil > \frac{|W|}{N}$, the last set $W_{\lceil \frac{|W|}{N} \rceil}$ has less than N elements.

[8] $U = \Lambda^{l+1}$ is because Λ^{l+1} contains the indices corresponding to N most significant elements in $\mathbf{\Phi}' \mathbf{r}^l$. Since $supp(\mathbf{u}) = \Omega \setminus T^l$ and $supp(\mathbf{z}) = T \cap T^k \cup \Gamma_\tau^k$ and also noting that $T^k \subseteq T^l$, we have $W = supp(\mathbf{u}) \cap supp(\mathbf{z}) = \Gamma_\tau^k \setminus T^l$.

Furthermore, we have

$$\begin{aligned}
&\langle \mathbf{\Phi}'\mathbf{r}^l, \mathbf{z}-\mathbf{x}^l\rangle \\
&= \langle \mathbf{\Phi}(\mathbf{z}-\mathbf{x}^l), \mathbf{r}^l\rangle && (128)\\
&= \frac{1}{2}\left(\|\mathbf{\Phi}(\mathbf{z}-\mathbf{x}^l)\|_2^2 + \|\mathbf{r}^l\|_2^2 - \|\mathbf{r}^l - \mathbf{\Phi}(\mathbf{z}-\mathbf{x}^l)\|_2^2\right) && (129)\\
&= \frac{1}{2}\left(\|\mathbf{\Phi}(\mathbf{z}-\mathbf{x}^l)\|_2^2 + \|\mathbf{r}^l\|_2^2 - \|\mathbf{\Phi}(\mathbf{x}-\mathbf{z}) + \mathbf{v}\|_2^2\right) && (130)\\
&= \frac{1}{2}\left(\|\mathbf{\Phi}(\mathbf{z}-\mathbf{x}^l)\|_2^2 + \|\mathbf{r}^l\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2\right), && (131)
\end{aligned}$$

where (130) uses the fact that $\mathbf{r}^l + \mathbf{\Phi}\mathbf{x}^l = \mathbf{y} = \mathbf{\Phi}\mathbf{x} + \mathbf{v}$.

In proving (125), we consider the following two cases.

- First, if $\|\mathbf{r}^l\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2 < 0$, (125) is true since $\|\mathbf{\Phi}'_{\Lambda^{l+1}}\mathbf{r}^l\|_2^2 \geq 0$.
- Next, if $\|\mathbf{r}^l\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2 \geq 0$, then using inequality $\frac{1}{2}(a+b) \geq \sqrt{ab}$ (with $a = \|\mathbf{\Phi}(\mathbf{z}-\mathbf{x}^l)\|_2^2$ and $b = \|\mathbf{r}^l\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2$), (131) becomes

$$\begin{aligned}
&\langle \mathbf{\Phi}'\mathbf{r}^l, \mathbf{z}-\mathbf{x}^l\rangle \\
&\geq \|\mathbf{\Phi}(\mathbf{z}-\mathbf{x}^l)\|_2 \sqrt{\|\mathbf{r}^l\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2}. && (132)
\end{aligned}$$

Noting that $\mathbf{z}-\mathbf{x}^l$ is supported on $\Gamma^k_\tau \cup T^l$, we have

$$\begin{aligned}
\|\mathbf{\Phi}(\mathbf{z}-\mathbf{x}^l)\|_2 &\geq \sqrt{1-\delta_{|\Gamma^k_\tau\cup T^l|}}\ \|\mathbf{z}-\mathbf{x}^l\|_2 \\
&\geq \sqrt{1-\delta_{|\Gamma^k_\tau\cup T^l|}}\ \|(\mathbf{z}-\mathbf{x}^l)_{\Omega\setminus T^l}\|_2 \\
&= \sqrt{1-\delta_{|\Gamma^k_\tau\cup T^l|}}\ \|\mathbf{z}_{\Omega\setminus T^l}\|_2, \qquad (133)
\end{aligned}$$

where (133) is due to $(\mathbf{x}^l)_{\Omega\setminus T^l} = \mathbf{0}$. Plugging (132) and (133) into (127), we have

$$\begin{aligned}
\|\mathbf{\Phi}'_{\Lambda^{l+1}}\mathbf{r}^l\|_2 &\geq \sqrt{\frac{1-\delta_{|\Gamma^k_\tau\cup T^l|}}{\lceil\frac{|\Gamma^k_\tau|}{N}\rceil}} \\
&\quad \times\sqrt{\|\mathbf{r}^l\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2}.
\end{aligned}$$

Combining these two cases, we obtain the desired result. ■

We are now ready to prove Proposition 2.3.

Proof: Using Lemma B.2 and B.3 and also noting that $\|\mathbf{r}^l - \mathbf{r}^{l+1}\|_2^2 = \|\mathbf{r}^l\|_2^2 - \|\mathbf{r}^{l+1}\|_2^2$, we have

$$\begin{aligned}
&\|\mathbf{r}^l\|_2^2 - \|\mathbf{r}^{l+1}\|_2^2 \\
&\geq \frac{1-\delta_{|\Gamma^k_\tau\cup T^l|}}{(1+\delta_N)\lceil\frac{|\Gamma^k_\tau|}{N}\rceil} \times \left(\|\mathbf{r}^l\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2\right), && (134)
\end{aligned}$$

which completes the proof. ■

Appendix C
Proof of Proposition 2.4

Proof: Subtracting both sides of (17) by $\|\mathbf{r}^l\|_2^2 - \|\mathbf{\Phi}_{T\setminus\Gamma^k_\tau}\mathbf{x}_{T\setminus\Gamma^k_\tau}\|_2^2$, we have

$$\begin{aligned}
&\|\mathbf{r}^{l+1}\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2 \\
&\leq \left(1 - \frac{1-\delta_{|\Gamma^k_\tau\cup T^l|}}{\lceil\frac{|\Gamma^k_\tau|}{N}\rceil(1+\delta_N)}\right) \\
&\quad \times(\|\mathbf{r}^l\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2). && (135)
\end{aligned}$$

Since $\frac{1-\delta_{|\Gamma^k_\tau\cup T^l|}}{\lceil\frac{|\Gamma^k_\tau|}{N}\rceil(1+\delta_N)} > 0$, we have

$$1 - \frac{1-\delta_{|\Gamma^k_\tau\cup T^l|}}{\lceil\frac{|\Gamma^k_\tau|}{N}\rceil(1+\delta_N)} \leq \exp\left(-\frac{1-\delta_{|\Gamma^k_\tau\cup T^l|}}{\lceil\frac{|\Gamma^k_\tau|}{N}\rceil(1+\delta_N)}\right).$$

Hence,

$$\begin{aligned}
&\|\mathbf{r}^{l+1}\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2 \\
&\leq \exp\left(-\frac{1-\delta_{|\Gamma^k_\tau\cup T^l|}}{\lceil\frac{|\Gamma^k_\tau|}{N}\rceil(1+\delta_N)}\right)(\|\mathbf{r}^l\|_2^2 \\
&\quad -\|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2), && (136)
\end{aligned}$$

and also

$$\begin{aligned}
&\|\mathbf{r}^{l+2}\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2 \\
&\leq \exp\left(-\frac{1-\delta_{|\Gamma^k_\tau\cup T^{l+1}|}}{\lceil\frac{|\Gamma^k_\tau|}{N}\rceil(1+\delta_N)}\right)(\|\mathbf{r}^{l+1}\|_2^2 \\
&\quad -\|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2), && (137)\\
&\vdots \\
&\|\mathbf{r}^{l'}\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2 \\
&\leq \exp\left(-\frac{1-\delta_{|\Gamma^k_\tau\cup T^{l'-1}|}}{\lceil\frac{|\Gamma^k_\tau|}{N}\rceil(1+\delta_N)}\right)(\|\mathbf{r}^{l'-1}\|_2^2 \\
&\quad -\|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2). && (138)
\end{aligned}$$

After some manipulations, we obtain

$$\begin{aligned}
&\|\mathbf{r}^{l'}\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2 \\
&\leq \prod_{i=l}^{l'-1}\exp\left(-\frac{1-\delta_{|\Gamma^k_\tau\cup T^l|}}{\lceil\frac{|\Gamma^k_\tau|}{N}\rceil(1+\delta_N)}\right) \\
&\quad \times(\|\mathbf{r}^l\|_2^2 - \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2). && (139)
\end{aligned}$$

Since $C_{\tau,l,l'} > 0$, it is clear from (**??**) that

$$\|\mathbf{r}^{l'}\|_2^2 \leq C_{\tau,l,l'}\|\mathbf{r}^l\|_2^2 + \|\mathbf{\Phi}_{\Gamma^k\setminus\Gamma^k_\tau}\mathbf{x}_{\Gamma^k\setminus\Gamma^k_\tau} + \mathbf{v}\|_2^2, \qquad (140)$$

which completes the proof. ■

References

[1] S. G. Mallat and Z. Zhang, "Matching pursuits with time-frequency dictionaries," *IEEE Trans. Signal Process.*, vol. 41, no. 12, pp. 3397–3415, Dec. 1993.

[2] J. A. Tropp, "Greed is good: Algorithmic results for sparse approximation," *IEEE Trans. Inform. Theory*, vol. 50, no. 10, pp. 2231–2242, Oct. 2004.

[3] J. A. Tropp and A. C. Gilbert, "Signal recovery from random measurements via orthogonal matching pursuit," *IEEE Trans. Inform. Theory*, vol. 53, no. 12, pp. 4655–4666, Dec. 2007.

[4] M. A. Davenport and M. B. Wakin, "Analysis of Orthogonal Matching Pursuit using the restricted isometry property," *IEEE Trans. Inform. Theory*, vol. 56, no. 9, pp. 4395–4401, Sep. 2010.

[5] J. Wang and B. Shim, "On the recovery limit of sparse signals using orthogonal matching pursuit," *IEEE Trans. Signal Process.*, vol. 60, no. 9, pp. 4973–4976, Sep. 2012.

[6] J. Ding, L. Chen, and Y. Gu, "Perturbation analysis of orthogonal matching pursuit," *IEEE Trans. Signal Process.*, vol. 61, no. 2, pp. 398–410, Jan. 2013.

[7] E. Livshitz, "On efficiency of orthogonal matching pursuit," 2010 [Online]. Available: http://arxiv.org/PS_cache/arxiv/pdf/1004/1004.3946v1.pdf.

[8] T. Zhang, "Sparse recovery with orthogonal matching pursuit under rip," *IEEE Trans. Inform. Theory*, vol. 57, no. 9, pp. 6215–6221, Sep. 2011.

[9] S. Foucart, "Stability and robustness of weak orthogonal matching pursuits," 2011 [Online]. Available: http://www.math.drexel.edu/~foucart/WOMMP.pdf.

[10] J. Wang and B. Shim, "Improved recovery bounds of orthogonal matching pursuit using restricted isometry property," *arXiv:1211.4293*, 2012.

[11] J. Wang, S. Kwon, and B. Shim, "Generalized orthogonal matching pursuit," *IEEE Trans. Signal Process.*, vol. 60, no. 12, pp. 6202–6216, Dec. 2012.

[12] E. Liu and V. N. Temlyakov, "The orthogonal super greedy algorithm and applications in compressed sensing," *IEEE Trans. Inform. Theory*, vol. 58, no. 4, pp. 2040–2047, Apr. 2012.

[13] S. Huang and J. Zhu, "Recovery of sparse signals using omp and its variants: convergence analysis based on rip," *Inverse Problems*, vol. 27, no. 3, pp. 035003, 2011.

[14] R. Maleh, "Improved rip analysis of orthogonal matching pursuit," *arXiv preprint arXiv:1102.4311*, 2011.

[15] E. J. Candès and T. Tao, "Decoding by linear programming," *IEEE Trans. Inform. Theory*, vol. 51, no. 12, pp. 4203–4215, Dec. 2005.

[16] R. Baraniuk, M. Davenport, R. DeVore, and M. Wakin, "A simple proof of the restricted isometry property for random matrices," *Constructive Approximation*, vol. 28, no. 3, pp. 253–263, 2008.

[17] D. L. Donoho and M. Elad, "On the stability of the basis pursuit in the presence of noise," *Signal Processing*, vol. 86, no. 3, pp. 511–532, 2006.

[18] E. J. Candès, "The restricted isometry property and its implications for compressed sensing," *Comptes Rendus Mathematique*, vol. 346, no. 9-10, pp. 589–592, 2008.

[19] D. Needell and R. Vershynin, "Signal recovery from incomplete and inaccurate measurements via regularized orthogonal matching pursuit," *IEEE J. Sel. Topics Signal Process.*, vol. 4, no. 2, pp. 310–316, Apr. 2010.

[20] D. Needell and J. A. Tropp, "Cosamp: Iterative signal recovery from incomplete and inaccurate samples," *Applied and Computational Harmonic Analysis*, vol. 26, no. 3, pp. 301–321, Mar. 2009.